SECTION A: PURE

Answer ALL questions.

1. The curve C has equation

$$y = 3x^4 - 8x^3 - 3$$

(a) Find (i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 2$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

(7 marks)

(a) (i) $\dfrac{dy}{dx} = 12x^3 - 24x^2$

(ii) $\dfrac{d^2y}{dx^2} = 36x^2 - 48x$

(b) when $x = 2$: $\dfrac{dy}{dx} = 12(2)^3 - 24(2)^2$

$\dfrac{dy}{dx} = 0$

$\therefore$ when $x = 2$, C has a stationary point.

(c) when $x = 2$: $\dfrac{d^2y}{dx^2} = 36(2)^2 - 48(2)$

$\dfrac{d^2y}{dx^2} = 48$

As $\dfrac{d^2y}{dx^2} > 0$ this point is a minimum point

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2. (a) Given that θ is small, use the small angle approximation for $\cos\theta$ to show that

$$1 + 4\cos\theta + 3\cos^2\theta \approx 8 - 5\theta^2.$$

(3)

Adele uses $\theta = 5°$ to test the approximation in part (a).

Adele's working is shown below.

Using my calculator, $1 + 4\cos(5°) + 3\cos^2(5°) = 7.962$, to 3 decimal places.
Using the approximation $8 - 5\theta^2$ gives $8 - 5(5)^2 = -117$
Therefore, $1 + 4\cos\theta + 3\cos^2\theta \approx 8 - 5\theta^2$ is not true for $\theta = 5°$.

(b) (i) Identify the mistake made by Adele in her working.

(ii) Show that $8 - 5\theta^2$ can be used to give a good approximation to $1 + 4\cos\theta + 3\cos^2\theta$ for an angle of size 5°.

(2)

(5 marks)

(a) $1 + 4\cos\theta + 3\cos^2\theta$

$\approx 1 + 4\left(1 - \frac{\theta^2}{2}\right) + 3\left(1 - \frac{\theta^2}{2}\right)^2$

$\approx 1 + 4 - 2\theta^2 + 3\left(1 - \theta^2 + \frac{\theta^4}{4}\right)$

$\approx 5 - 2\theta^2 + 3 - 3\theta^2 + \frac{3}{4}\theta^4$

$\approx 8 - 5\theta^2$ [as θ^4 is very small]

(b) (i) The small angle approximations only work when θ is in radians. Adele has used degrees.

(ii) $180° = \pi$ radians

$5° = \frac{\pi}{36}$ radians

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$$8 - 5\left(\frac{\pi}{36}\right)^2 = 7.962 \text{ (to 3dp)}$$

Therefore $8 - 5\theta^2$ is a good approximation for an angle of size $5°$.

Alternative to part (a)

$$1 + 4\cos\theta + 3\cos^2\theta$$

$$= 1 + 4\cos\theta + 3(1 - \sin^2\theta)$$

$$= 1 + 4\cos\theta + 3 - 3\sin^2\theta$$

$$= 4 + 4\cos\theta - 3\sin^2\theta$$

$$\approx 4 + 4\left(1 - \frac{\theta^2}{2}\right) - 3\theta^2$$

$$\approx 4 + 4 - 2\theta^2 - 3\theta^2$$

$$\approx 8 - 5\theta^2$$

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3. A circle C has equation

$$x^2 + y^2 - 4x + 10y = k,$$

where k is a constant.

(a) Find the coordinates of the centre of C.

(2)

(b) State the range of possible values for k.

(2)

(4 marks)

(a) $x^2 - 4x + y^2 + 10y = k$

$(x-2)^2 - 4 + (y+5)^2 - 25 = k$

$(x-2)^2 + (y+5)^2 = k + 29$

$\Rightarrow$ the centre of C is $(2, -5)$

(b) $k + 29 > 0$

$\therefore \quad k > -29$

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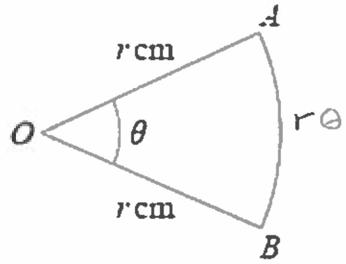
4.



The diagram shows a sector AOB of a circle with centre O and radius r cm. The angle AOB is θ radians. The area of the sector AOB is 11 cm^2.

Given that the perimeter of the sector is 4 times the length of the arc AB, find the exact value of r.

(4 marks)

$P = r + r + r\theta$

$P = 2r + r\theta$

As $P = 4(r\theta)$

$\Rightarrow 2r + r\theta = 4r\theta$

$3r\theta = 2r$

$\therefore \theta = \frac{2}{3}$ rads

$A = \frac{1}{2}r^2\theta$ and $A = 11$

$\Rightarrow \frac{1}{2}r^2\theta = 11$

$\frac{1}{2}r^2\left(\frac{2}{3}\right) = 11$

$r^2 = 33$

$\therefore r = \sqrt{33}$

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5. $$f(x) = -3x^3 + 8x^2 - 9x + 10, x \in \mathbb{R}.$$

(a) (i) Calculate f(2).

(ii) Write f (x) as a product of two algebraic factors.

(3)

Using the answer to part (a) (ii),

(b) prove that there are exactly two real solutions to the equation

$$-3y^6 + 8y^4 - 9y^2 + 10 = 0,$$

(2)

(c) deduce the number of real solutions, for $7\pi \leq \theta < 10\pi$, to the equation

$$3\tan^3\theta - 8\tan^2\theta + 9\tan\theta - 10 = 0.$$

(1)

(6 marks)

(a)(i) $f(2) = -3(2)^3 + 8(2)^2 - 9(2) + 10$

$f(2) = 0$

(ii) As $f(2) = 0$, $(x-2)$ is a factor of $f(x)$

$\Rightarrow f(x) = (x-2)(-3x^2 + 2x - 5)$

[working : $-3x^3 + \underbrace{6x^2 + 2x^2}_{8x^2} \underbrace{- 4x - 5x}_{-9x} + 10$]

(b) $-3y^6 + 8y^4 - 9y^2 + 10 = 0$

Let $x = y^2$

$\Rightarrow -3x^3 + 8x^2 - 9x + 10 = 0$

$(x-2)(-3x^2 + 2x - 5) = 0$

When $x - 2 = 0$

$x = 2$

$\Rightarrow \quad y^2 = 2$

$y = \pm\sqrt{2}$

when $-3x^2 + 2x - 5 = 0$

$b^2 - 4ac$

$= (2)^2 - 4(-3)(-5)$

$= -56$

As $-56 < 0$, there are no real roots.

As $x = y^2$, there are also no real roots for y.

$\therefore$ The only real solutions to the equation are $y = \pm\sqrt{2}$.
So there are exactly two real solutions.

(c) Only solutions are when $\tan\theta = 2$.

As $\tan\theta$ repeats every π radians, there are 3 solutions between 7π and 10π.

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6. Complete the table below. The first one has been done for you.

For each statement below you must state if it is always true, sometimes true or never true, giving a reason in each case.

Statement	Always True	Sometimes True	Never True	Reason
The quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has 2 real roots.		✓		It only has 2 real roots when $b^2 - 4ac > 0$ When $b^2 - 4ac = 0$ it has 1 real root and when $b^2 - 4ac < 0$ it has 0 real roots.
(i) When a real value of x is substituted into $x^2 - 6x + 10$ the result is positive. **(2)**	✓			$= (x-3)^2 - 9 + 10$ $= (x-3)^2 + 1$ As $(x-3)^2 > 0$ $(x-3)^2 + 1$ is always positive $\therefore$ Always true
(ii) If $ax > b$ then $x > \frac{b}{a}$ **(2)**		✓		True for a is positive. Not true for a is negative. e.g. when $a = 2$ $2x > b \Rightarrow x > \frac{b}{2}$ when $a = -2$ $-2x > b \Rightarrow x < \frac{b}{-2}$ $\therefore$ Sometimes true
(iii) The difference between consecutive square numbers is odd. **(2)**	✓			$(n^2+1) - n^2$ $= n^2 + 2n + 1 - n^2$ $= 2n + 1$ As $2n$ is even, $2n+1$ is odd $\therefore$ Always true

(6 marks)

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7. (a) Use the binomial expansion, in ascending powers of x, to show that

$$\sqrt{(4-x)} = 2 - \frac{1}{4}x + kx^2 + \ldots$$

where k is a rational constant to be found.

(4)

A student attempts to substitute $x = 1$ into both sides of this equation to find an approximate value for $\sqrt{3}$.

(b) State, giving a reason, if the expansion is valid for this value of x.

(1)

(5 marks)

(a) $(4-x)^{\frac{1}{2}}$

$= \left[4\left(1-\frac{1}{4}x\right)\right]^{\frac{1}{2}}$

$= 4^{\frac{1}{2}}\left(1-\frac{1}{4}x\right)^{\frac{1}{2}}$

$= 2\left(1-\frac{1}{4}x\right)^{\frac{1}{2}}$

$= 2\left[1 + \left(\frac{1}{2}\right)\left(-\frac{1}{4}x\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{4}x\right)^2}{2}\right]$

$= 2\left[1 - \frac{1}{8}x - \frac{1}{8}\left(\frac{1}{16}x^2\right)\right]$

$= 2\left[1 - \frac{1}{8}x - \frac{1}{128}x^2\right]$

$= 2 - \frac{1}{4}x - \frac{1}{64}x^2$ where $k = -\frac{1}{64}$

(b) For $(1+x)^n$, $|x| < 1$

For $(4-x)^{\frac{1}{2}}$, $|-x| < 4 \Rightarrow |x| < 4$

Therefore, as x is only valid for $|x| < 4$, $x = 1$ can't be used.

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8. The depth of water, D metres, in a harbour on a particular day is modelled by the formula

$$D = 5 + 2\sin(30t)°, \quad 0 \le t < 24,$$

where t is the number of hours after midnight.

A boat enters the harbour at 6:30 a.m. and it takes 2 hours to load its cargo. The boat requires the depth of water to be at least 3.8 metres before it can leave the harbour.

(a) Find the depth of the water in the harbour when the boat enters the harbour.

(1)

(b) Find, to the nearest minute, the earliest time the boat can leave the harbour.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(4)

(5 marks)

(a) When $t=0$: $D = 5 + 2\sin(30\times6.5)°$

$D = 4.48\text{m}$ (3sf)

(b) $D \geq 3.8$ $\Rightarrow$ earliest time when $D = 3.8$

$5 + 2\sin(30t) = 3.8$

$\sin(30t) = -\frac{3}{5}$ $\quad 0 \leq 30t < 720$

$30t = -36.9$

$30t = 216.9$

$30t = 323.1, 683.1, 576.9$

$\therefore$ $30t = 216.9, 323.1, 683.1, 576.9$

$t = 7.23, 10.77, 22.77, 19.23$

6.30 am + 2 hrs = 8.30 am

$\therefore$ earliest $t = 10.77$ $\Rightarrow$ earliest time is 10:47 am

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9. The curve C has equation $y = 2x^3 + 5$.

The curve C passes through the point $P(1, 7)$.

Use differentiation from first principles to find the value of the gradient of the tangent to C at P.

(5 marks)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{(2(1+h)^3 + 5) - (2(1)^3 + 5)}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{2(1 + 3(1)^2h + 3(1)h^2 + h^3) + 5 - (2+5)}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{2 + 6h + 6h^2 + 2h^3 + 5 - 7}{h}$$

$$f'(1) = \lim_{h \to 0} 6 + 6h + 2h^2$$

As $h \to 0$, $f'(1) \to 6$

$\therefore$ the gradient of the tangent to C at P is 6.

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10. A company decides to manufacture a soft drinks can with a capacity of 500 ml.

The company models the can in the shape of a right circular cylinder with radius r cm and height h cm. In the model they assume that the can is made from a metal of negligible thickness.

(a) Prove that the total surface area, S cm^2, of the can is given by

$$S = 2\pi r^2 + \frac{1000}{r}$$

(3)

Given that r can vary,

(b) find the dimensions of a can that has minimum surface area.

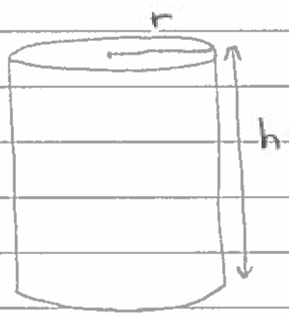
(5)

(c) With reference to the shape of the can, suggest a reason why the company may choose not to manufacture a can with minimum surface area.

(1)

(9 marks)

(a)



$V = \pi r^2 h$

As $V = 500$, $\pi r^2 h = 500$ ①

$S = 2(\pi r^2) + 2\pi r h$ ②

① : $h = \frac{500}{\pi r^2}$

sub into ② : $S = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$

$\therefore S = 2\pi r^2 + \frac{1000}{r}$

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(b) $S = 2\pi r^2 + 1000r^{-1}$

$\frac{dS}{dr} = 4\pi r - 1000r^{-2}$

When $\frac{dS}{dr} = 0$: $4\pi r - 1000r^{-2} = 0$

$4\pi r - \frac{1000}{r^2} = 0$

$4\pi r^3 - 1000 = 0$

$r^3 = \frac{1000}{4\pi}$

$r = 4.30$ cm (3sf)

$\pi r^2 h = 500$

$h = \frac{500}{\pi r^2}$

when $r = 4.30...$: $h = \frac{500}{\pi(4.30...)^2}$

$h = 8.60$ cm (3sf)

(c) The height of the can will be approximately the same size as the diameter of the can.
This would make the can more difficult to hold when drinking out of it.

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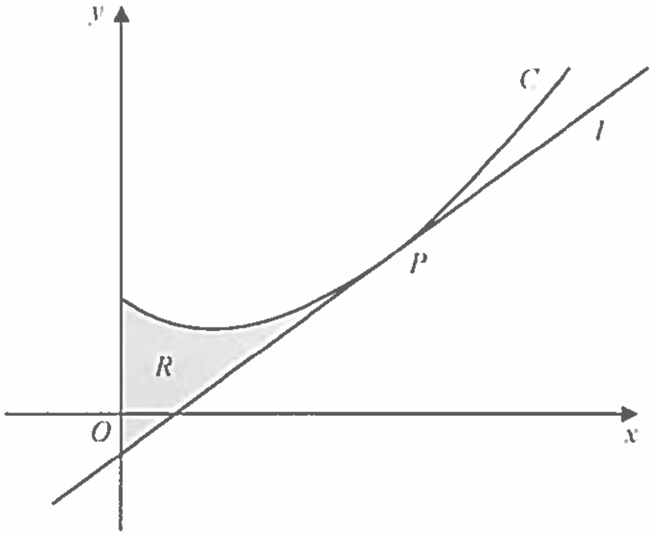
11.



The diagram shows a sketch of the curve C with equation

$$y = 5x^{\frac{3}{2}} - 9x + 11, \quad x \geq 0$$

The point P with coordinates (4, 15) lies on C.

The line l is the tangent to C at the point P.

The region R, shown shaded in the diagram, is bounded by the curve C, the line l and the y-axis.

Show that the area of R is 24, making your method clear.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(10 marks)

$\frac{dy}{dx} = \frac{15}{2} x^{\frac{1}{2}} - 9$

when $x = 4$: $\frac{dy}{dx} = \frac{15}{2}(4)^{\frac{1}{2}} - 9$

$\therefore m_T = 6$

$y - y_1 = m(x - x_1)$

$y - 15 = 6(x - 4)$

$\therefore$ equation of l is $y = 6x - 9$

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Area between curve and x-axis, bounded at $x=0$ and $x=4$:

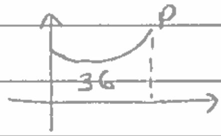
$$\int_0^4 5x^{\frac{3}{2}} - 9x + 11 \, dx$$

$$= \left[2x^{\frac{5}{2}} - \frac{9}{2}x^2 + 11x\right]_0^4$$

$$= \left[2(4)^{\frac{5}{2}} - \frac{9}{2}(4)^2 + 11(4)\right] - \left[2(0)^{\frac{5}{2}} - \frac{9}{2}(0)^2 + 11(0)\right]$$

$$= [64 - 72 + 44] - [0]$$

$$= 36$$



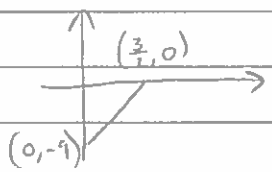
l crosses the x-axis when $y=0$: $0 = 6x - 9$

$x = \frac{3}{2}$

$\Rightarrow \left(\frac{3}{2}, 0\right)$

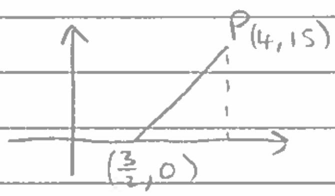
l crosses the y-axis when $x=0$: $y = -9$

$\Rightarrow (0, -9)$



Area of triangle $= \frac{1}{2} \times \frac{3}{2} \times 9$

$= \frac{27}{4}$



Area of triangle $= \frac{1}{2} \times \left(4 - \frac{3}{2}\right) \times 15$

$= \frac{75}{4}$

Area of $R = 36 - \frac{75}{4} + \frac{27}{4}$

$= 24$

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SECTION B: STATISTICS

Answer ALL questions.

12. Naasir is playing a game with two friends. The game is designed to be a game of chance so that the probability of Naasir winning each game is $\frac{1}{3}$.

Naasir and his friends play the game 15 times.

(a) Find the probability that Naasir wins

(i) exactly 2 games,

(ii) more than 5 games.

(3)

Naasir claims he has a method to help him win more than $\frac{1}{3}$ of the games. To test this claim, the three of them played the game again 32 times and Naasir won 16 of these games.

(b) Stating your hypotheses clearly, test Naasir's claim at the 5% level of significance.

(4)

(7 marks)

(a) $X \sim B(15, \frac{1}{3})$

(i) $P(X=2) = 0.0599$ (3sf)

(ii) $P(X>5) = 1 - P(X \leq 5)$

$= 1 - 0.6183...$

$= 0.382$ (3sf)

(b) Let Y = the number of games Naasir wins

p = probability of winning

$H_0: p = \frac{1}{3}$

$H_1: p > \frac{1}{3}$

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$Y \sim B\left(32, \frac{1}{3}\right)$

$P(Y \geq 16) = 1 - P(Y \leq 15)$

$= 1 - 0.9623$

$= 0.0377$ (3sf)

As $0.0377 < 0.05$ we reject H_0.

There is significant evidence to support Naasir's claim that his method helps him win more than $\frac{1}{3}$ of the games.

13. Given that $P(A) = 0.35$, $P(B) = 0.45$ and $P(A \cap B) = 0.13$,

(a) find $P(A' | B')$, **(2)**

(b) explain why the events A and B are not independent. **(1)**

The event C has $P(C) = 0.20$.

The events A and C are mutually exclusive and the events B and C are statistically independent.

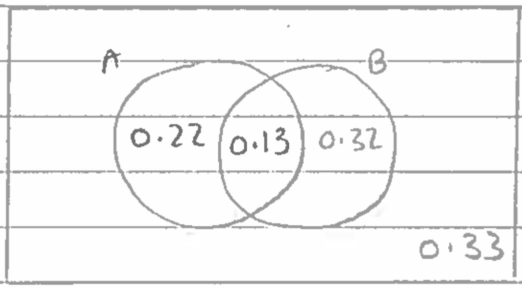
(c) Draw a Venn diagram to illustrate the events A, B and C, giving the probabilities for each region. **(5)**

(d) Find $P([B \cup C]')$ **(2)**

(10 marks)

$0.35 - 0.13$

$= 0.22$



$0.45 - 0.13$

$= 0.32$

$1 - (0.22 + 0.13 + 0.32) = 0.33$

(a) $P(A' | B') = \dfrac{P(A' \cap B')}{P(B')}$

$= \dfrac{0.33}{(0.22 + 0.33)}$

$= 0.6$

(b) $P(A) \times P(B) = 0.35 \times 0.45 = 0.1575$

$P(A \cap B) = 0.13$

As $0.1575 \neq 0.13$

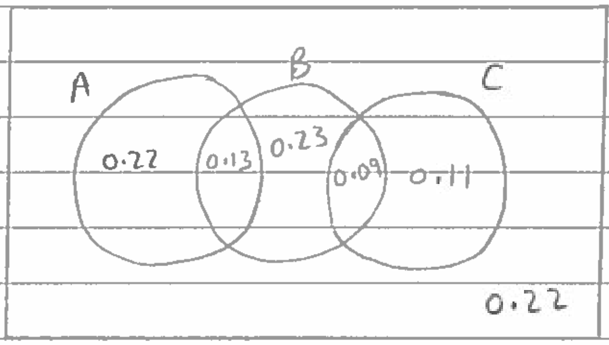
$P(A)\,P(B) \neq P(A \cap B)$

Therefore A and B are not independent.

(c) As B and C are independent : $P(B)P(C) = P(B \cap C)$

$\Rightarrow P(B \cap C) = 0.45 \times 0.20 = 0.09$

$0.45 - (0.13 + 0.09)$
$= 0.23$



$0.20 - 0.09$
$= 0.11$

$1 - (0.22 + 0.13 + 0.23 + 0.09 + 0.11) = 0.22$

(d) $P([B \cup C]') = 0.22 + 0.22$

$= 0.44$

SECTION C: MECHANICS

Answer ALL questions.

14. A small ball is projected vertically upwards from a point A which is 19.6 m above the ground. The ball strikes the ground, for the first time, 4 s later.

The motion of the ball is modelled as that of a particle moving freely under gravity.

(a) Use the model to find the speed of the ball as it hits the ground for the first time.

(3)

The ball rebounds from the ground with a vertical speed of 14.7 m s^{-1} and next comes to instantaneous rest at the point B.

(b) Use the model to find the height of B above the ground.

(2)

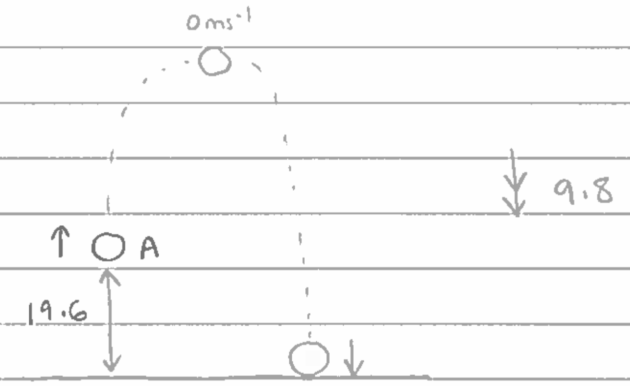
In a refined model of the motion of the ball, the effect of air resistance is included and this refined model is now used to find the speed of the ball as it hits the ground for the first time

(c) How would this new value of the speed of the ball as it hits the ground for the first time compare with the value found using the initial model in part (a)?

(1)

(6 marks)

(a)



From A to ground $(\uparrow)$

$s = -19.6$ $\qquad s = vt - \frac{1}{2}at^2$

$u =$

$v = ?$ $\qquad (-19.6) = v(4) - \frac{1}{2}(-9.8)(4)^2$

$a = -9.8$

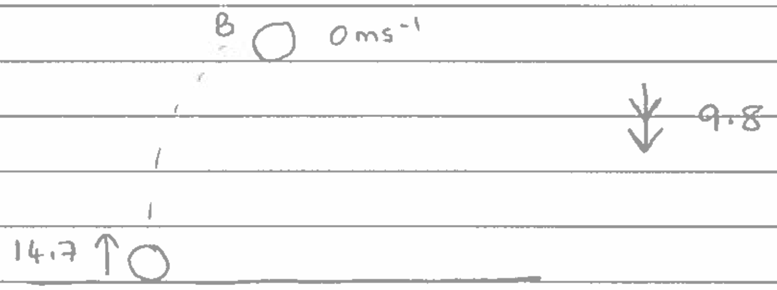
$t = 4$ $\qquad -19.6 = 4v + 78.4$

$4v = -98$

$V = -24.5$

Therefore the speed of the ball as it hits the ground is 24.5 ms^{-1}.

(b)



From ground to B (↑)

$s = ?$
$u = 14.7$
$v = 0$
$a = -9.8$
$t =$

$v^2 = u^2 + 2as$

$(0)^2 = (14.7)^2 + 2(-9.8)s$

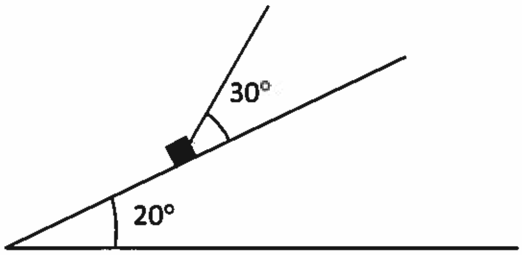
$19.6s = 216.09$

$s = 11.0$ (3sf)

Therefore the height of B above the ground is 11.0 m

(c) Air resistance would slow down the ball quicker than acceleration due to gravity alone.
Therefore this new value of the speed of the ball as it hits the ground would be less than the value found in part (a).

15.



A small box of mass 3 kg moves on a rough plane which is inclined at an angle of 20° to the horizontal. The box is pulled up a line of greatest slope of the plane using a rope which is attached to the box. The rope makes an angle of 30° with the plane, as shown in the diagram. The rope lies in the vertical plane which contains a line of greatest slope of the plane. The coefficient of friction between the box and the plane is 0.3. The tension in the rope is 25 N.

The box is modelled as a particle, the rope is modelled as a light inextensible string and air resistance is ignored.

Using the model,

(a) find the acceleration of the box.

(7)

(b) Suggest one improvement to the model that would make it more realistic.

(1)

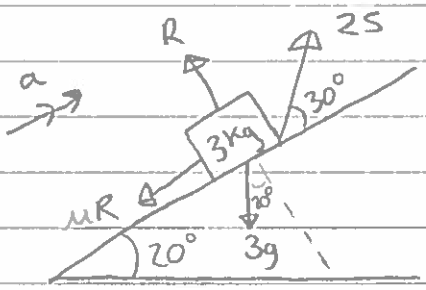
The rope now breaks and the box slows down and comes to rest.

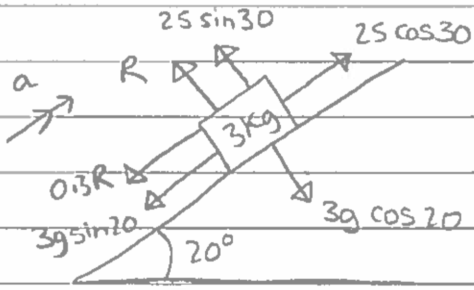
(c) Show that, after the box comes to rest, it immediately starts to move down the plane.

(3)

(11 marks)

(a)





Perp (↑): $F = ma$

$R + 25\sin 30 - 3g\cos 20 = 0$

$R = 3g\cos 20 - 25\sin 30$

Para ($\nearrow$) : $F = ma$

$$25\cos 30 - 3g\sin 20 - 0.3R = 3a$$

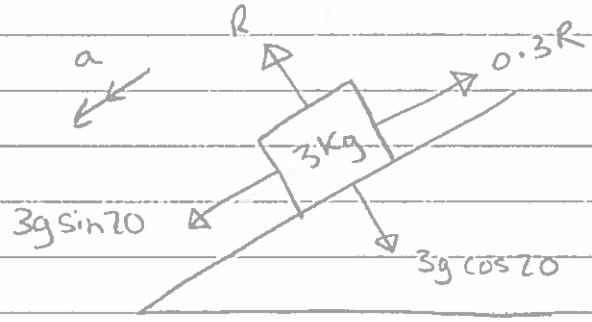
$$25\cos 30 - 3g\sin 20 - 0.3(3g\cos 20 - 25\sin 30) = 3a$$

$$3a = 7.057\ldots$$

$$a = 2.35\ ms^{-2}\ (3sf)$$

(b) To make the model more realistic we could include air resistance.

(c) When the rope breaks, there is no tension. When the box stops moving up the plane, the component of weight will cause the box to want to move down the plane. Therefore the direction of friction will be up the plane.



Perp ($\uparrow$) : $F = ma$

$$R - 3g\cos 20 = 0$$

$$R = 3g\cos 20$$

Para ($\swarrow$): $F = ma$

$$3g\sin 20 - 0.3R = 3a$$

$$3g\sin 20 - 0.3(3g\cos 20) = 3a$$

$$3a = 1.767\ldots$$

$$a = 0.589\ ms^{-2}\ (3sf)$$

As $a > 0$, the box will move down the plane.