U6 A Level Maths PURE MOCK
April 2019
Time: 2 hours  Total Marks: 100

You must have:  Mathematical Formulae and Statistical Tables, Calculator

Instructions
- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Fill in your name at the top of this page and the name of your teacher.
- Answer all questions and ensure that your answers to parts of questions are clearly labelled.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit. Answers found from the calculator without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information
- A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
- There are 14 questions. The total mark for this paper is 100.
- The marks for each question are shown in brackets
  – use this as a guide as to how much time to spend on each question.

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

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Grade
1. \[ g(x) = \frac{2x + 5}{x - 3}, \quad x \geq 5. \]

(a) Find \( gg(5) \).

\[ \begin{align*}
\text{(a)} & \quad g(5) = \frac{2(5) + 5}{5 - 3} \\
& \quad = 7.5 \\
\end{align*} \]

\[ \begin{align*}
gg(5) = g(7.5) = \frac{2(7.5) + 5}{(7.5) - 3} \\
& \quad = \frac{40}{8} \\
& \quad = 5
\end{align*} \]

(b) \[ g(x) = \frac{2(x - 3) + 11}{x - 3}, \quad x \geq 5 \]

\[ \begin{align*}
& \quad = 2 + \frac{11}{x - 3}, \quad x > 5 \\
\end{align*} \]

\[ \text{range: } 2 \leq g(x) \leq \frac{15}{2} \]
(c) \[ y = \frac{2x + 5}{x - 3} \]

\[ x = \frac{2y + 5}{y - 3} \]

\[ x(y - 3) = 2y + 5 \]

\[ xy - 3x = 2y + 5 \]

\[ xy - 2y = 3x + 5 \]

\[ y(x - 2) = 3x + 5 \]

\[ y = \frac{3x + 5}{x - 2} \]

\[ \therefore g^{-1}(x) = \frac{3x + 5}{x - 2}, \quad 2 < x \leq \frac{15}{2} \]
2. Relative to a fixed origin \( O \),

the point \( A \) has position vector \((2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})\),
the point \( B \) has position vector \((4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\),
and the point \( C \) has position vector \((a\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})\), where \( a \) is a constant and \( a < 0 \).

\( D \) is the point such that \( \overrightarrow{AB} = \overrightarrow{BD} \).

(a) Find the position vector of \( D \). 

\( \overrightarrow{AC} = 4 \),

(b) find the value of \( a \).

\( 5 \) marks

\[
(a) \quad \overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}
\]

\[
\overrightarrow{BD} = \begin{pmatrix} x - (4) \\ y - (-2) \\ z - (3) \end{pmatrix} = \begin{pmatrix} x - 4 \\ y + 2 \\ z - 3 \end{pmatrix}
\]

\[
\overrightarrow{AB} = \overrightarrow{BD}
\]

\[
\therefore \begin{pmatrix} x - 4 \\ y + 2 \\ z - 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}
\]

\[
\Rightarrow x = 6, \quad y = -7, \quad z = 10
\]

The position vector of \( D \) is \( \begin{pmatrix} 6 \\ -7 \\ 10 \end{pmatrix} \).
(b) \( \overrightarrow{AC} = \begin{pmatrix} a-2 \\ 2 \\ 2 \end{pmatrix} \)

\[ |\overrightarrow{AC}| = \sqrt{(a-2)^2 + 2^2 + 2^2} \]

\[ = \sqrt{a^2 - 4a + 12} \]

\[ |\overrightarrow{AC}| = 4 \]

\[ \therefore \sqrt{a^2 - 4a + 12} = 4 \]

\[ a^2 - 4a + 12 = 16 \]

\[ a^2 - 4a - 4 = 0 \]

\[ a = 2 + 2\sqrt{2} \quad \text{or} \quad a = 2 - 2\sqrt{2} \]

As \( a < 0 \), \( a = 2 - 2\sqrt{2} \)
3. A sequence of numbers $a_1, a_2, a_3, \ldots$, is defined by

$$a_1 = 3,$$

$$a_{n+1} = \frac{a_n - 3}{a_n - 2}, \quad n \in \mathbb{N}.$$ 

(a) Find $\sum_{r=1}^{100} a_r$. 

(b) Hence find $\sum_{r=1}^{100} a_r + \sum_{r=1}^{99} a_r$.

\[a_2 = \frac{a_1 - 3}{a_1 - 2}\]

\[= \frac{(3) - 3}{(3) - 2}\]

\[= \frac{0}{1}\]

\[= 0\]

\[a_3 = \frac{a_2 - 3}{a_2 - 2}\]

\[= \frac{(0) - 3}{(0) - 2}\]

\[= \frac{-3}{-2}\]

\[= \frac{3}{2}\]

\[a_4 = \frac{a_3 - 3}{a_3 - 2}\]

\[= \frac{\left(\frac{3}{2}\right) - 3}{\left(\frac{3}{2}\right) - 2}\]

\[= \frac{-\frac{3}{2}}{-\frac{1}{2}}\]

\[= 3\]

$a_5 = 0$, $a_6 = \frac{3}{2}$ etc.
Sequence: \([\frac{3}{2}, 0, \frac{3}{2}, 3, 0, \frac{3}{2}, \ldots, 3]\)

\[
\sum_{r=1}^{100} a_r = 33 \left( \frac{3 + 0 + \frac{3}{2}}{2} \right) + 3
\]

\[= \frac{303}{2}\]

(b) \[
\sum_{r=1}^{100} a_r + \sum_{r=1}^{99} a_r = \frac{303}{2} + 33 \left( \frac{3 + 0 + \frac{3}{2}}{2} \right)
\]

\[= 300\]
4. Given \( y = x(2x + 1)^4 \), show that

\[
\frac{dy}{dx} = (2x + 1)^4 (4x + B)
\]

where \( n, A \) and \( B \) are constants to be found.

\[
y = xc (2x + 1)^4
\]

\[
\frac{dy}{dx} = (1)(2x + 1)^4 + (xc) (4(2x + 1)^3 \times 2)
\]

\[
= (2x + 1)^4 + 8x (2x + 1)^3
\]

\[
= (2x + 1)^3 (2x + 1 + 8x)
\]

\[
= (2x + 1)^3 (10x + 1) \quad \text{with } n = 4, A = 10 \text{ and } B = 1
\]

(4 marks)
5. Given that \( a \) is a positive constant and
\[
\int_{a}^{2a} \frac{t+1}{t} \, dt = \ln 7,
\]
show that \( a = \ln k \), where \( k \) is a constant to be found. (4 marks)

\[
\int_{a}^{2a} \frac{t+1}{t} \, dt = \int_{a}^{2a} \left(1 + \frac{1}{t}\right) \, dt
\]
\[
= \left[ t + \ln t \right]_{a}^{2a}
\]
\[
= \left[ 2a + \ln(2a) \right] - \left[ a + \ln(a) \right]
\]
\[
= a + \ln(2a) - \ln(a)
\]
\[
= a + \ln \left( \frac{2a}{a} \right)
\]
\[
= a + \ln(2)
\]

\[
\int_{a}^{2a} \frac{t+1}{t} \, dt = \ln 7
\]
\[
\therefore \ a + \ln(2) = \ln 7
\]
\[
\Rightarrow a = \ln 7 - \ln 2
\]
\[
\Rightarrow a = \ln \left( \frac{7}{2} \right) \quad \text{with} \quad k = \frac{7}{2}
\]
6. \( f(x) = -3x^3 + 8x^2 - 9x + 10, x \in \mathbb{R}. \)

(a) (i) Calculate \( f(2) \).

(ii) Write \( f(x) \) as a product of two algebraic factors.

Using the answer to part (a) (ii),

(b) prove that there are exactly two real solutions to the equation

\[-3y^6 + 8y^4 - 9y^2 + 10 = 0,\]

(2) (5 marks)

(a) (i) \( f(2) = -3(2)^3 + 8(2)^2 - 9(2) + 10 \)

\[= 0 \]

(ii) As \( f(2) = 0 \), \( (x - 2) \) is a factor of \( f(x) \)

\[\Rightarrow f(x) = (x - 2)(-3x^2 + 2x - 5) \]

[working: \(-3x^3 + 6x^2 + 2x^2 - 4x - 5x + 10\)]

(b) Let \( x = y^2 \)

\[f(x) = f(y^2)\]

\[(x - 2)(-3x^2 + 2x - 5) = (y^2 - 2)(-3(y^2)^2 + 2y^2 - 5)\]

When \( f(y^2) = 0 \)

\[(y^2 - 2)(-3(y^2)^2 + 2y^2 - 5) = 0\]
\[ y^2 - 2 = 0 \quad \text{or} \quad -3(y) + 2y^2 - 5 = 0 \]

\[ y = \pm \sqrt{2} \quad \quad b^2 - 4ac \]

\[ = (2)^2 - 4(-3)(-5) \]

\[ = -56 \]

As \(-56 < 0\), there are no real solutions.

Therefore, there are only two real solutions to this equation: \( y = \sqrt{2} \) or \( y = -\sqrt{2} \).
7. (i) Solve, for $0 \leq x < \frac{\pi}{2}$, the equation

\[ 4 \sin x = \sec x. \]  \hspace{1cm} (4)

(ii) Solve, for $0 \leq \theta < 360^\circ$, the equation

\[ 5 \sin \theta - 5 \cos \theta = 2, \]

giving your answers to one decimal place.

(Solutions based entirely on graphical or numerical methods are not acceptable.)  \hspace{1cm} (5)

(9 marks)

\[
4 \sin x = \sec x
\]

\[
4 \sin x = \frac{1}{\cos x}
\]

\[
4 \sin x \cos x = 1
\]

\[
2 \sin x \cos x = \frac{1}{2}
\]

\[
\sin 2x = \frac{1}{2}
\]

$0 \leq x < \frac{\pi}{2} \implies 0 \leq 2x < \pi$

1st. \quad $2x = \sin^{-1} \left( \frac{1}{2} \right) = \frac{\pi}{6}$

2nd. \quad $2x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

\[ 2x = \frac{\pi}{6}, \quad \frac{5\pi}{6} \]

\[ \therefore \quad x = \frac{\pi}{12}, \quad \frac{5\pi}{12} \]
\[ S \sin \theta - S \cos \theta = 2 \quad (1) \]

\[ \sin(A - B) = \sin A \cos B - \cos A \sin B \]

\[ R \sin(\theta - B) = R \sin \theta \cos B - R \cos \theta \sin B \quad (2) \]

Comparing (1) and (2): \[ R \cos B = S \]
\[ R \sin B = S \]

\[ \cos B = \frac{S}{R} \quad \sin B = \frac{S}{R} \]

\[ R = \sqrt{S^2 + S^2} = S \sqrt{2} \]

\[ \tan B = \frac{S}{S} \Rightarrow B = 45^\circ \]

\[ S \sin \theta - S \cos \theta = S \sqrt{2} \sin(\theta - 45^\circ) \]

\[ \Rightarrow S \sqrt{2} \sin(\theta - 45^\circ) = 2 \]

\[ \sin(\theta - 45^\circ) = \frac{\sqrt{2}}{S} \quad -45^\circ \leq \theta - 45^\circ \leq 315^\circ \]

1st: \[ \theta - 45^\circ = \sin^{-1}\left(\frac{\sqrt{2}}{S}\right) = 16.4^\circ \]

2nd: \[ \theta - 45^\circ = 180^\circ - 16.4^\circ = 163.6^\circ \]

\[ \theta - 45^\circ = 16.4^\circ, 163.6^\circ \]

\[ \theta = 61.4^\circ, 208.6^\circ \]
8. \[ f(x) = \ln(2x - 5) + 2x^2 - 30, \quad x > 2.5. \]

(a) Show that \( f(x) = 0 \) has a root \( \alpha \) in the interval \([3.5, 4]\).

(2 marks)

A student takes 4 as the first approximation to \( \alpha \).

Given \( f(4) = 3.099 \) and \( f'(4) = 16.67 \) to 4 significant figures,

(b) apply the Newton-Raphson procedure once to obtain a second approximation for \( \alpha \),

giving your answer to 3 significant figures.

(2 marks)

(c) Show that \( \alpha \) is the only root of \( f(x) = 0 \).

(2 marks)

(6 marks)

\[
\begin{align*}
(a) \quad f(3.5) &= \ln(2(3.5) - 5) + 2(3.5)^2 - 30 \\
&= -4.806852819 \ldots \\

f(4) &= \ln(2(4) - 5) + 2(4)^2 - 30 \\
&= 3.098612289 \ldots \\

\text{As there's a change of sign and } f(x) \text{ is continuous, the root } \alpha \text{ lies between 3.5 and 4.}
\end{align*}
\]

\[
\begin{align*}
(b) \quad x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= \left(4\right) - \frac{f(4)}{f'(4)} \\
&= 4 - \frac{3.099}{16.67} \\
&= 2.81 (3sf) \\
\therefore \quad \alpha &\approx 2.81 (3sf)
\end{align*}
\]
(c) \[ \ln(2x-5) + 2x^2 - 30 = 0 \]

\[ \ln(2x-5) = 30 - 2x^2 \]

The graphs of \( y = 30 - 2x^2 \) and \( y = \ln(2x-5) \) only meet at one coordinate. Therefore, \( \ln(2x-5) = 30 - 2x^2 \) only has one solution and so \( f(x) = 0 \) has only one root.
9. An archer shoots an arrow.

The height, $H$ metres, of the arrow above the ground is modelled by the formula

$$H = 1.8 + 0.4d - 0.002d^2, \quad d \geq 0,$$

where $d$ is the horizontal distance of the arrow from the archer, measured in metres.

Given that the arrow travels in a vertical plane until it hits the ground,

(a) find the horizontal distance travelled by the arrow, as given by this model. \hspace{1cm} (3)

(b) With reference to the model, interpret the significance of the constant 1.8 in the formula. \hspace{1cm} (1)

(c) Write $1.8 + 0.4d - 0.002d^2$ in the form

$$A - B(d - C)^2$$

where $A$, $B$ and $C$ are constants to be found. \hspace{1cm} (3)

It is decided that the model should be adapted for a different archer.

The adapted formula for this archer is

$$H = 2.1 + 0.4d - 0.002d^2, \quad d \geq 0.$$

Hence, or otherwise, find, for the adapted model,

(d) (i) the maximum height of the arrow above the ground.

(ii) the horizontal distance, from the archer, of the arrow when it is at its maximum height. \hspace{1cm} (2)

(9 marks)

\begin{align*}
(a) \quad & \text{When } H = 0, \quad 1.8 + 0.4d - 0.002d^2 = 0 \\
\quad & 0.002d^2 - 0.4d - 1.8 = 0 \\
\quad & d = \frac{-(-0.4) \pm \sqrt{(-0.4)^2 - 4(0.002)(1.8)}}{2(0.002)} \\
\quad & \Rightarrow d = 704m \quad (3SF) \\
(b) \quad & 1.8 \text{ is the height above the ground the arrow is when held by the archer (when } d = 0). 
\end{align*}
(c) \[ 1.8 + 0.4d - 0.002d^2 \]
\[ = -0.002d^2 + 0.4d + 1.8 \]
\[ = -0.002 \left[ d^2 - 200d \right] + 1.8 \]
\[ = -0.002 \left[ (d-100)^2 - 100^2 \right] + 1.8 \]
\[ = -0.002 (d-100)^2 + 20 + 1.8 \]
\[ = 21.8 - 0.002 (d-100)^2 \]

where \( A = 21.8 \), \( B = 0.02 \) and \( c = 100 \)

(d) \[ H = 2.1 + 0.4d - 0.002d^2 \]
\[ = -0.002 (d-100)^2 + 20 + 2.1 \]
\[ = 27.1 - 0.002 (d-100)^2 \]

(i) \[ \therefore \text{max height} = 27.1 \text{ m} \]

(ii) \[ \therefore \text{when max height: } d = 100 \text{ m} \]
10. In a controlled experiment, the number of microbes, \( N \), present in a culture \( T \) days after the start of the experiment, were counted.

\( N \) and \( T \) are expected to satisfy a relationship of the form

\[ N = aT^b, \]

where \( a \) and \( b \) are constants.

(a) Show that this relationship can be expressed in the form

\[ \log_{10} N = m \log_{10} T + c, \]

giving \( m \) and \( c \) in terms of the constants \( a \) and/or \( b \).

(b) Use the information provided to estimate the number of microbes present in the culture 3 days after the start of the experiment.

(c) Explain why the information provided could not reliably be used to estimate the day when the number of microbes in the culture first exceeds 1,000,000.

(d) With reference to the model, interpret the value of the constant \( a \).

(9 marks)
(a) \[ N = a T^b \]

\[ \log_{10} N = \log_{10} (a T^b) \]
\[ \log_{10} N = \log_{10} a + \log_{10} T^b \]
\[ \log_{10} N = \log_{10} a + b \log_{10} T \]
\[ \log_{10} N = b \log_{10} T + \log_{10} a \]

\[ \therefore \log_{10} N = m \log_{10} T + c \]

where \( m = b \) and \( c = \log_{10} a \)

(b) \[ m = \frac{1.85}{0.8} = 2.3125 \]

\[ \Rightarrow b = 2.3125 \]

\[ c = 1.8 \]

\[ \Rightarrow \log_{10} a = 1.8 \]

\[ a = 10^{1.8} \]

\[ T = 3 \]

\[ \therefore N = (10^{1.8})^{(2.3125)} \]

\[ N \approx 800 \text{ (3sf)} \]
(c) If \( N > 1,000,000 \)

then \( \log_{10}N > \log_{10}1,000,000 \)

so \( \log_{10}N > 6 \)

Our data from the experiment only goes up to 4.6 for \( \log_{10}N \) and so values outside of this range would not be reliable (extrapolation).

(d) \( a \) represents the number of microbes present 1 day after the start of the experiment.
11. A company decides to manufacture a soft drinks can with a capacity of 500 ml.

The company models the can in the shape of a right circular cylinder with radius \( r \) cm and height \( h \) cm. In the model they assume that the can is made from a metal of negligible thickness.

(a) Prove that the total surface area, \( S \) cm\(^2\), of the can is given by

\[
S = 2\pi r^2 + \frac{1000}{r}.
\]  
(3)

Given that \( r \) can vary,

(b) find the dimensions of a can that has minimum surface area.

(c) With reference to the shape of the can, suggest a reason why the company may choose not to manufacture a can with minimum surface area.

\[ V = \pi r^2 h \Rightarrow \pi r^2 h = 500 \]  
(1)

\[ S = 2\pi r^2 + 2\pi rh \]  
(2)

Sub (1) into (2):

\[ S = 2\pi r^2 + 2\pi r \left( \frac{500}{\pi r^2} \right) \]

\[ \therefore S = 2\pi r^2 + \frac{1000}{r} \]
(b) \[ S = 2\pi r^2 + 1000r^{-1} \]

\[ \frac{dS}{dr} = 0 \]

\[ 4\pi r - 1000r^{-2} = 0 \]

\[ 4\pi r - \frac{1000}{r^2} = 0 \]

\[ 4\pi r^3 - 1000 = 0 \]

\[ r^3 = \frac{1000}{4\pi} \]

\[ r = 4.30127... \]

\[ r = 4.30 \text{ (3sf)} \]

Sub into (1): \[ h = \frac{500}{\pi (4.30127...)^2} \]

\[ h = 8.60 \text{ (3sf)} \]

(c) The shape of the can is almost a cube, when most drinks cans are taller than wider to make them easier to hold and drink out of.
12. Kayden claims that $3^x \geq 2^x$.

(i) Determine whether Kayden’s claim is always true, sometimes true or never true, justifying your answer.

(ii) Prove that $\sqrt{3}$ is an irrational number.

(i) \[ \text{when } x = 1 : \quad 3 > 1 \quad \text{✓ not true.} \]

\[ \text{when } x = 0 : \quad 1 = 1 \quad \text{✓ true.} \]

\[ \text{when } x = -1 : \quad \frac{1}{3} < \frac{1}{2} \quad \text{✗ not true.} \]

Therefore, Kayden’s claim is sometimes true.

(ii) Assume $\sqrt{3}$ is a rational number.

So $\sqrt{3} = \frac{p}{q}$ where $p$ and $q$ are integers, $q \neq 0$ and HCF of $p$ and $q$ is 1.

\[ \Rightarrow p = \sqrt{3}q \]

\[ p^2 = 3q^2 \quad \text{(1)} \]

Therefore, as $p^2$ is divisible by 3, $p$ is also divisible by 3:

\[ \Rightarrow p = 3c \quad \text{(2)} \text{ where } c \text{ is an integer.} \]

Sub (2) into (1):

\[ (3c)^2 = 3q^2 \]

\[ q^2 = 3c^2 \]

Therefore, as $q^2$ is divisible by 3, $q$ is also divisible by 3.
As $p$ and $q$ are both divisible by 3, the HCF of $p$ and $q$ is not 1.

This contradicts our original assumption. Therefore $\sqrt{3}$ is an irrational number.
13. A curve $C$ has parametric equations
\[ x = 3 + 2 \sin t, \quad y = 4 + 2 \cos 2t, \quad 0 \leq t < 2\pi. \]

(a) Show that all points on $C$ satisfy $y = 6 - (x - 3)^2$.  \(2\)

(b) (i) Sketch the curve $C$.
(ii) Explain briefly why $C$ does not include all points of $y = 6 - (x - 3)^2$, $x \in \mathbb{R}$. \(3\)

The line with equation $x + y = k$, where $k$ is a constant, intersects $C$ at two distinct points.

(c) State the range of values of $k$, writing your answer in set notation. \(5\)

\[ (a) \quad x = 3 + 2 \sin t \]

\[ x - 3 = 2 \sin t \]

\[ (x - 3)^2 = 4 \sin^2 t \quad (1) \]

\[ y = 4 + 2 \cos 2t \]

\[ y = 4 + 2 (1 - 2 \sin^2 t) \]

\[ y = 4 + 2 - 4 \sin^2 t \]

\[ y = 6 - 4 \sin^2 t \quad (2) \]

Sub 1 into 2: $y = 6 - (x - 3)^2$
(b) (i) \(0 \leq t < 2\pi \Rightarrow -1 \leq \sin t \leq 1 \Rightarrow 1 \leq x \leq 5\)

\(\cos 2t = \frac{1 - \cos 2t}{2} \Rightarrow -1 \leq \cos 2t \leq 1 \Rightarrow 2 \leq y \leq 6\)

\[ y = 6 - (x^2 - 6x + 9) \Rightarrow y = -x^2 + 6x - 3 \]

when \(x = 1\) : \(y = 2\)
when \(x = 5\) : \(y = 2\)

(ii) The values of \(t\) are restricted, which means the values of \(x\) and \(y\) are restricted as well.

\(1 \leq x \leq 5\) and \(2 \leq y \leq 6\)

(c) \(y = k - x \quad \text{(1)} \quad y = 6 - (x - 3)^2 \quad \text{(2)}\)

\((1) = (2) : \quad k - x = 6 - (x - 3)^2\)

\[ k - x = -x^2 + 6x - 3 \]

\[ x^2 - 7x + (k + 3) = 0 \]

\[ b^2 - 4ac > 0 \Rightarrow (-7)^2 - 4(1)(k + 3) > 0 \]

\[ 49 - 4k - 12 > 0 \Rightarrow k < \frac{37}{4} \]
The lowest value of $k$ at which $y = k - x$ and $y = 6 - (x - 3)^2$ intersect twice is at the point $(5, 2)$.

$$k = x + y \implies k = 5 + 2 = 7$$

Therefore $k \geq 7$.

Set notation: $k = \left\{ k : 7 \leq k < \frac{37}{4} \right\}$
14. (a) Express $\frac{1}{P(11 - 2P)}$ in partial fractions.

A population of meerkats is being studied.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{22} P(11 - 2P), \quad t \geq 0, \quad 0 < P < 5.5,$$

where $P$, in thousands, is the population of meerkats and $t$ is the time measured in years since the study began.

Given that there were 1000 meerkats in the population when the study began,

(b) determine the time taken, in years, for this population of meerkats to double,

(c) show that

$$P = \frac{A}{B + Ce^{\frac{t}{1}}},$$

where $A$, $B$ and $C$ are integers to be found.

(a) \[ \frac{1}{P(11 - 2P)} = \frac{A}{P} + \frac{B}{11 - 2P} \]

\[ \frac{1}{P(11 - 2P)} = \frac{A(11 - 2P) + B(P)}{P(11 - 2P)} \]

\[ \Rightarrow \quad 1 = A(11 - 2P) + B(P) \]

Let $P = 0$ : \[ 11A = 1 \quad \Rightarrow \quad A = \frac{1}{11} \]

Let $P = \frac{11}{2}$ : \[ \frac{11}{2}B = 1 \quad \Rightarrow \quad B = \frac{2}{11} \]

\[ \therefore \quad \frac{1}{P(11 - 2P)} = \frac{1}{11P} + \frac{2}{11(11 - 2P)} \]
\[ \frac{dP}{dt} = \frac{1}{22} \rho (11 - 2P) \]

\[ \int \frac{1}{\rho (11 - 2P)} \, dP = \int \frac{1}{22} \, dt \]

\[ \frac{1}{11} \int \left( \frac{1}{P} + \frac{2}{11 - 2P} \right) \, dP = \int \frac{1}{22} \, dt \]

\[ \frac{1}{11} \left[ \ln P - \ln (11 - 2P) \right] = \frac{1}{22} \, t + C_1 \]

\[ \ln P - \ln (11 - 2P) = \frac{1}{2} t + C_2 \]

When \( t = 0 \), \( P = 1 \), let \( C_2 = \ln K \)

\[ \ln (1) - \ln (9) = \ln K \]

\[ \ln K = \ln \frac{1}{9} \]

\[ \therefore \ln P - \ln (11 - 2P) = \frac{1}{2} t + \ln \left( \frac{1}{9} \right) \]

When population is 2000, \( P = 2 \):

\[ \ln (2) - \ln (7) = \frac{1}{2} \, t + \ln \left( \frac{1}{9} \right) \]

\[ \ln \left( \frac{2}{7} \right) = \frac{1}{2} \, t + \ln \left( \frac{1}{9} \right) \]
\[ \frac{1}{2} t = \ln \left( \frac{2}{9} \right) - \ln \left( \frac{1}{9} \right) \]

\[ \frac{1}{2} t = \ln \left( \frac{\frac{1}{9}}{\frac{3}{9}} \right) \]

\[ \frac{1}{2} t = \ln \left( \frac{1}{3} \right) \]

\[ t = 2 \ln \left( \frac{\frac{1}{3}}{\frac{1}{9}} \right) \]

\[ t = 1.8889 \ldots \]

Time taken in years for the population to double is 1.89 (3sf)

\[ \begin{align*}
\ln P - \ln (11-2P) & = \frac{1}{2} t + \ln \frac{1}{9} \\
\ln \left( \frac{P}{11-2P} \right) & = \frac{1}{2} t + \ln \frac{1}{9} \\
\ln \left( \frac{P}{11-2P} \right) - \ln \left( \frac{1}{9} \right) & = \frac{1}{2} t \\
\ln \left( \frac{9P}{11-2P} \right) & = \frac{1}{2} t
\end{align*} \]
\[
\frac{qP}{(11-2p)} = e^{\frac{1}{2}t}
\]

\[
qP = (11-2p) e^{\frac{1}{2}t}
\]

\[
qP = 11e^{\frac{1}{2}t} - 2Pe^{\frac{1}{2}t}
\]

\[
qP + 2P e^{\frac{1}{2}t} = 11e^{\frac{1}{2}t}
\]

\[
P(9+2e^{\frac{1}{2}t}) = 11e^{\frac{1}{2}t}
\]

\[
P = \frac{11e^{\frac{1}{2}t}}{9+2e^{\frac{1}{2}t}}
\]

\[
P = \frac{11}{q e^{-\frac{1}{2}t} + 2}
\]

\[
\Rightarrow P = \frac{11}{2 + q e^{-\frac{1}{2}t}} \text{ where } A = 11, B = 2 \text{ and } C = q
\]